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A GLOBAL VERSION OF THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS

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1. Introduction

In [4], Tonti gave necessary and sufficient conditions for certain differential expressions (namely those expressions which we call "source equations"; for the definition see below or [3]) to be *locally* the Euler equation of some variational problem. In this paper we consider the corresponding global problem. To state the results we need some definitions.

In what follows, $\pi: E \to W$ will be some fixed differentiable fibration i.e., (E, π, W) is a fibre bundle, E and W are smooth manifolds and π has everywhere maximal rank. A *variational problem*, or *Lagrangian*, on π is an operator $\mathfrak{L}$ which assigns to each smooth (local) section S: $W \to E$ of π an n-form ($n = \dim(W)$) $\mathfrak{L}(S)$ on the domain of S such that, for each x in the domain of S, $(\mathfrak{L}(S))(x)$ only depends (smoothly) on the value of S, and on a finite number of its derivatives, in x.

A *source equation* on a bundle π is an operator $\mathfrak{S}$ which assigns to each smooth (local) section S: $W \to E$ of π and each x in the domain of S an element $(\mathfrak{S}(S))(x) \in (\mathrm{Ker}(d\pi)_{S(x)})^* \otimes \Lambda^n(T_x^*(W))$, which only depends (smoothly) on the value of S, and a finite number of its derivatives in x.

A source equation $\mathfrak{S}$ is the Euler equation of the Lagrangian $\mathfrak{L}$ if for each bounded (i.e., having compact closure) oriented open $U \subset W$, and each smooth 1-parameter family of local sections S_t of π with the properties:

(i) for each $t \in (-\varepsilon, +\varepsilon)$, $\overline{U} \subset$ interior of the domain of S_t, and

(ii) $S_t(x)$ is independent of t if $x \notin U$, we have

$$\frac{d}{dt}\int_{\overline{U}} \mathfrak{L}(S_t)\bigg|_{t=0} = \int_{\overline{U}} \langle \mathfrak{S}(S_t)(x), \dot{S}_t(x)\rangle\bigg|_{t=0},$$

where $\dot{S}_t(x)$ denotes the tangent vector of the curve $t \to S_t(x)$; this tangent vector is in $\mathrm{Ker}(d\pi)_{S_t(x)}$ so for each x, $\langle \mathfrak{S}(S_t)(x), \dot{S}_t(x)\rangle|_{t=0} \in \Lambda^n(T_x^*(W))$. Hence on both left- and right-hand side there is an n-form under the integral sign. The integral is defined because U is oriented and bounded.

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The above definition of "Euler equation" is obtained by adapting the usual one to the "coordinate-free language" which one has to use when dealing with arbitrary differentiable bundles; see also the introduction of [3].

The inverse problem of the calculus of variations is concerned with the question how to decide whether a given source equation is the Euler equation of a variational problem. If a source equation satisfies the condition of Tonti, a corresponding Lagrangian can be constructed locally (local refers here to E). If the bundle is sufficiently simple, e.g., $\pi: E \to W$ is a vector bundle, then the Tonti condition is enough to guarantee the existence of a (global) Lagrangian. We want to determine for which bundles π the Tonti condition is necessary and sufficient to guarantee the existence of a Lagrangian corresponding to a given source equation (the Tonti condition is always necessary). A source equation is said to be *locally variational* if it satisfies everywhere the Tonti condition, and to be *variational* if it is the Euler equation of some variational problem. Our main result is

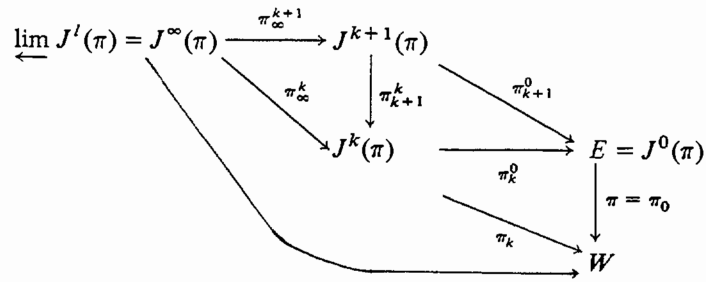
Theorem. *The vector space of locally variational source equations modulo the variational source equations is canonically isomorphic with* $H^{n+1}(E; \mathbf{R})$, $n = \dim(W)$.

The paper is organized as follows. In §2 we introduce the space of ∞-jets of (local) sections of our differentiable fibration π and define smooth functions, vector fields and differential forms on this jet space. These definitions were also given in [3], but are repeated here to make the paper self-contained. In §3 we consider Lie derivatives and various types of exterior derivatives for differential forms on this jet space. In §4 we prove some local exactness theorems for these differential forms, and relate in some case the lack of global exactness to the real cohomology of E as in the theorem of de Rham. In §5 we finally identify Lagrangians and source equations as certain types of differential forms on our ∞-jet space, and deduce the main theorem from the results in §4.

2. Differential geometry of infinite-jet spaces

A number of the results in this section can also be found in [3]; the present presentation is more complete, and in some points the proofs are simplified.

Let $\pi: E \to W$ again be a differentiable fibration, i.e., π is a C^∞ map (unless stated otherwise, everything in this paper is C^∞) which has everywhere maximal rank and is a bundle projection in the topological sense. $J^k(\pi)$, the space of k-jets of local sections of π, is defined as follows: $J^k(\pi)$ is the set of equivalence classes of pairs (w, S), $w \in W$ and S local cross section of π defined on a neighborhood of w; $(w, S) \sim (w', s')$ if and only if $w = w'$ and all the derivatives of S and S' up to and including order k are equal in $w = w'$. $J^k(\pi)$ has, in the obvious way, the structure of a smooth manifold. There are canonical projections π_k^l: $J^k(\pi) \to J^l(\pi)$ whenever $0 \leq l \leq k$, and π_k: $J^k(\pi) \to W$. Note that $J^0(\pi) = E$, so π_0: $E \to W$ equals π. The space of ∞-jets of local sections, which is denoted by $J^\infty(\pi)$, is, as set, the inverse limit of the system $\{J^k(\pi), \pi_l^k\}$; the induced projections are denoted by π_∞^k, π_∞. We define on $J^\infty(\pi)$ the inverse limit topology: if $s \in J^\infty(\pi)$, and U is a subset of $J^\infty(\pi)$ containing s, then U is a neighborhood of s if and only if there are some $k \in \mathbf{N}$ and neighborhood U_k of $\pi_\infty^k(s)$ in $J^k(\pi)$ such that $(\pi_\infty^k)^{-1}(U_k) \subset U$. The description of all the projections above can be visualized in the following diagram:



The *differentiable structure of* $J^\infty(\pi)$ is determined by specifying the C^∞ functions on $J^\infty(\pi)$. A function f: $J^\infty(\pi) \to \mathbf{R}$ is said to be C^∞, or to belong to $C^\infty(J^\infty(\pi))$, if for each $s \in J^\infty(\pi)$ there are a $k \in \mathbf{N}$, a neighborhood U_k of $\pi_\infty^k(s)$ in $J^k(\pi)$ and a C^∞ function f_k: $U_k \to \mathbf{R}$ such that $f|(\pi_\infty^k)^{-1}(U_k) = f_k \circ \pi_\infty^k$.

Proposition (2.1). *$J^\infty(\pi)$ is paracompact, and each open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of $J^\infty(\pi)$ admits a smooth partition of unity, i.e., for each such covering $\mathfrak{U}$ there is a set of smooth functions φ_i: $J^\infty(\pi) \to \mathbf{R}$, $i \in I$, such that $\{$support $(\varphi_i)\}_{i \in I}$ is locally finite, (support $\varphi_i) \subset U_i$, and $\sum_{i \in I} \varphi_i(s) = 1$ for each $s \in J^\infty(\pi)$.*

Proof. We first show that $J^\infty(\pi)$ is separable. For this, let $Q_k \subset J^k(\pi)$ be a countable dense set for each $k \in \mathbf{N}$; Q_k exists because $J^k(\pi)$, being a manifold, is separable. We can clearly choose some countable $Q \subset J^\infty(\pi)$ such that for each k, $\pi_\infty^k(Q) \supset Q_k$. From the definition of the topology of $J^\infty(\pi)$ it follows that Q is dense. The points of Q are denoted by $q_1, q_2, \ldots$.

For each $q_i \in Q$ we consider the following basis of neighborhoods:

$$B_{ij} = \{ s \in J^\infty(\pi) | \rho_j(\pi_\infty^j(s), \pi_\infty^j(q_i)) < 2^{-j} \},$$

where ρ_j, for each $j \in \mathbf{N}$, is a metric on $J^j(\pi)$; these metrics are supposed to be compatible in the sense that whenevery $x, y \in J^k(\pi)$,

$$\rho_k(x, y) = \min_{x' \in (\pi_{k+1}^k)^{-1}(x),\, y' \in (\pi_{k+1}^k)^{-1}(y)} \rho_{k+1}(x', y').$$

Clearly, $\{B_{ij}\}_{i,j \in \mathbf{N}}$ is a basis for the topology of $J^\infty(\pi)$. Hence each open covering $\mathcal{U}$ of $J^\infty(\pi)$ admits a refinement $\mathcal{V} = \{V_l\}_{l \in \mathbf{N}}$ with each $V_l \in \{B_{ij}\}_{i,j \in \mathbf{N}}$, say $V_l = B_{\alpha(l),\beta(l)}$, and such that for each l, $\overline{V}_l \subset U_i$ for some $i \in I$. In order to obtain from $\mathcal{V}$ a locally finite refinement, we first define

$$V_{l,m} = \left\{ s \in J^\infty(\pi) \mid \rho_{\beta(l)}\left(\pi_\infty^{\beta(l)}(s), \pi_\infty^{\beta(l)}(q_{\alpha(l)})\right) < 2^{-\beta(l)} - 2^{-m} \right\}$$

for each l, $m \in \mathbf{N}$; the locally finite refinement $\mathcal{W} = \{W_i\}_{i \in I}$ of $\mathcal{V}$ is now defined by

$$W_1 = V_1,\ W_2 = V_2 \setminus \overline{V}_{1,2}\ W_3 = V_3 \setminus \left(\overline{V}_{1,3} \cup \overline{V}_{2,3}\right), \text{ etc.}$$

Hence $J^\infty(\pi)$ is paracompact (this proof was essentially copied from Lang [2]). Since for each $l \in \mathbf{N}$, there are an integer $\gamma(l)$ and an open subset $\tilde{W}_l$ of $J^{\gamma(l)}(\pi)$ such that $(\pi_\infty^{\gamma(l)})^{-1}(\tilde{W}_l) = W_l$, there is a $\Psi_l \in C^\infty(J^\infty(\pi))$ such that $\Psi_l(s) > 0$ if $s \in W_l$, and $\Psi_1(s) = 0$ if $s \notin W_1$. Let now σ: $\mathbf{N} \to I$ be a function such that for each $l \in \mathbf{N}$, $\overline{W}_l \subset U_{\sigma(1)}$. Then we define for $i \in I$:

$$\varphi_i(s) = \left(\sum_{\{j \in \mathbf{N} \mid \sigma(j) = i\}} \Psi_j(s) \right) \cdot \left(\sum_{j \in \mathbf{N}} \Psi_j(s) \right)^{-1}.$$

This is the required partition of unity.

Definition 2.2. A *vector field* on $J^\infty(\pi)$ is a linear derivation on $C^\infty(J^\infty(\pi))$, i.e., a vector field X is a map $C^\infty(J^\infty(\pi)) \to C^\infty(J^\infty(\pi))$ such that $X(\alpha f + \beta g) = \alpha. X(f) + \beta. X(g)$ for $f, g \in C^\infty(J^\infty(\pi))$ and $\alpha, \beta \in \mathbf{R}$,

$$X(f.g) = f.X(g) + g.X(f).$$

$\mathcal{X}(J^\infty(\pi))$ denotes the vector space of vector fields on $J^\infty(\pi)$. If $A \subset J^\infty(\pi)$ is a subset, and X, $Y \in \mathcal{X}(J^\infty(\pi))$, then we say that X and Y are equal on A if for each $f \in C^\infty(J^\infty(\pi))$, $X(f)|A = Y(f)|A$. $\mathcal{X}(A)$ is the set of equivalence classes of $\mathcal{X}(J^\infty(\pi))$ under the equivalence relation $\sim_A$: $X \sim_A Y$ if and only if X and Y are equal on A. $\mathcal{X}(A)$ can be interpreted as the set of vector fields defined on A and extendable to $J^\infty(\pi)$. If $a \in J^\infty(\pi)$, $\mathcal{X}(\{a\})$ is also denoted by $T_a(J^\infty(\pi))$. Notice that the elements of $T_a(J^\infty(\pi))$ are just the linear derivations from $C^\infty(J^\infty(\pi))$ to $\mathbf{R}$ (in a). For $X \in \mathcal{X}(J^\infty(\pi))$ its equivalence class in $T_a(J^\infty(\pi))$ is denoted by $X(a)$.

Examples and definitions (2.3). Let $\overline{X}$ be a vector field on W; we shall associate to it in a canonical way a vector field X on $J^\infty(\pi)$; X is called the *total vector field* of $\overline{X}$. To do so we have to define $(X(f))(s)$ for each

$f \in C^\infty(J^\infty(\pi))$ and $s \in J^\infty(\pi)$. For this we choose a local cross section S of π, defined in a neighborhood of $\pi_\infty(s)$ and such that the ∞-jet of S in $\pi_\infty(s)$ equals s. Then we define $(X(f))(s)$ as $(\overline{X}(f \circ S))(\pi_\infty(s))$. It is not hard to verify that, for each $s \in J^\infty(\pi)$, the subset of $T_s(J^\infty(\pi))$, consisting of those elements which can be represented by total vector fields, is a linear subspace of $T_s(J^\infty(\pi))$ whose dimension equals that of W. We denote this subspace by H_s and call its elements *horizontal vectors*.

We say that a vector $v \in T_s(J^\infty(\pi))$ is *vertical* if for each $f\colon W \to \mathbf{R}$, $v(f \circ \pi_\infty) = 0$. The vertical vectors in $T_s(J^\infty(\pi))$ form a vector space, denoted by V_s. Clearly $H_s \cap V_s = \{0\}$ and $H_s + V_s = T_s(J^\infty(\pi))$.

It should be mentioned that the existence of the *canonical splitting* of $T_s(J^\infty(\pi))$ as $H_s \oplus V_s$, which cannot be constructed for $J^k(\pi)$, is the basis for many of our constructions; the main reason to work on $J^\infty(\pi)$, instead of $J^k(\pi)$, is the need for such a canonical splitting.

Lemma (2.4). *For each $s \in J^\infty(\pi)$, $T_s(J^\infty(\pi))$ is canonically isomorphic with the inverse limit of $\{T_{\pi^k_\infty(s)}(J^k(\pi)), d\pi^l_k\}_{k \geq l; l,k \in \mathbf{N}}$.*

Proof. First we define maps $d\pi^k_\infty\colon T_s(J^\infty(\pi)) \to T_{\pi^k_\infty(s)}(J^k(\pi))$ as follows: for $X \in T_s(J^\infty(\pi))$ and $f\colon J^k(\pi) \to \mathbf{R}$ a smooth function, $((d\pi^k_\infty)(X))(f) = X(f \circ \pi^k_\infty)$. Then clearly $d\pi^l_k \circ d\pi^k_\infty = d\pi^l_\infty$, $l \leq k$. We have to prove that if $X \in T_s(J^\infty(\pi))$ and $X \neq 0$, there is a k such that $d\pi^k_\infty(X) \neq 0$. For this we take a smooth function $f\colon J^\infty(\pi) \to \mathbf{R}$ such that $X(f) \neq 0$. Then by definition there are a $k \in \mathbf{N}$, a neighborhood U_k of $\pi^k_\infty(s)$ in $J^\infty(\pi)$ and a smooth function $\tilde{f}$ on U_k such that $f|(\pi^k_\infty)^{-1}(U_k) = \tilde{f} \circ \pi^k_\infty$. From this it follows that $d\pi^k_\infty(X) \neq 0$.

Finally we have to prove that for each sequence $X_1, X_2, \cdots, X_i \in T_{\pi^i_\infty(s)}(J^i(\pi))$ such that $d\pi^l_k(X_k) = X_l$ whenever $l \leq k$, there is an $X \in T_s(J^\infty(\pi))$ such that $\pi^k_\infty(X) = X_k$ for all k. For this we construct a sequence $\tilde{X}_1, \tilde{X}_2, \ldots$ of vector fields on $J^1(\pi), J^2(\pi), \ldots$ such that for each k, $\tilde{X}_k(\pi^k_\infty(s)) = X_k$ and for each $l \leq k$ and $s_k \in J^k(\infty)$, $\tilde{X}_l(\pi^l_k(s_k)) = d\pi^l_k(X_k(s_k))$. Such a sequence can easily be constructed by induction, starting with $\tilde{X}_1, \tilde{X}_2, \ldots$. The vector field $\tilde{X}$ on $J^\infty(\pi)$ is now defined by $(\tilde{X}(f)(s')) = (\tilde{X}_k(\tilde{f}))(\pi^k_\infty(s'))$ whenever $s' \in J^\infty(\pi)$, $\tilde{f}\colon J^k(\pi) \to \mathbf{R}$ is a smooth function and, on a neighborhood of s', $f = \tilde{f} \circ \pi^k_\infty$. Then $X = \tilde{X}(s)$ is the required vector in $T_s(J^\infty(\pi))$.

Lemma (2.5). *A vector field X on $J^\infty(\pi)$ determines a sequence of mappings $X_i\colon J^\infty(\pi) \to T(J^i(\pi))$, $i = 0, 1, 2, \ldots : X_i(s) = d\pi^i_\infty(X(s))$. These mappings X_i satisfy*

1. $X_i(s) \in T_{\pi^i_\infty(s)}(J^i(\pi))$,
2. $d\pi^{i-1}_i(X_i(s)) = X_{i-1}(s)$,

3. *for each smooth* f: $J^i(\pi) \to \mathbf{R}$, *the function* $J^\infty(\pi) \ni s \to (X_i(s))(f)$ *is smooth on* $J^\infty(\pi)$.

On the other hand, any sequence of mappings X_i: $J^\infty(\pi) \to T(J^i(\pi))$ satisfying the above conditions 1, 2 and 3 uniquely determines a vector field on $J^\infty(\pi)$.

Proof. Trivial consequence of Lemma (2.4).

Remarks and definitions (2.6). Let $\bar{X}$ be a vector field on E such that whenever $\pi(e) = \pi(e')$, $d\pi(\bar{X}(e)) = d\pi(\bar{X}(e')) = \pi(\bar{X})(\pi(e))$ (this last equation defines the vector field $\pi(\bar{X})$ on W). $\bar{X}$ induces a vector field on $J^\infty(\pi)$ in the following way:

For each i, $\bar{X}$ induces a vector field X_i on $J^i(\pi)$ such that $X_i(s)$, $s \in J^i(\pi)$, is the tangent vector of the curve

$$t \to \left(\mathfrak{D}_{\bar{X},t} \circ S \circ \mathfrak{D}_{\pi(\bar{X}),-t'}\mathfrak{D}_{\pi(\bar{X}),t}(\pi_i(s))\right)_k \in J^i(\pi),$$

where S is a local section of π representing s (hence defined in a neighborhood of $\pi_i(s)$), and $\mathfrak{D}_{\bar{X}\circ t} \circ S \circ \mathfrak{D}_{\pi(\bar{X}),-t}$ consequently is a section defined on a neighborhood of $\mathfrak{D}_{\pi(\bar{X}),t}(\pi_i(s))$ and thereby defining an element of $J^i(\pi)$.

Next we define $\tilde{X}_i$: $J^\infty(\pi) \to T(J^i(\pi))$ by $X_i \circ \pi^i_\infty$; these maps $\tilde{X}_i$ satisfy conditions 1, 2 and 3 in (2.5) and hence determine a vector field X on $J^\infty(\pi)$. A vector field on $J^\infty(\pi)$ which can be obtained in this way is said to be *integrable*; in case also $\pi(\bar{X}) \equiv 0$, it is said to be *vertical integrable*. The original vector field $\bar{X}$ on E, for which $\pi(\bar{X})$ on W could be defined, is called a *symmetry of* π.

Sometimes we need vector fields which are somewhat more general than integrable ones, namely *deformations*. A vector field X on $J^\infty(\pi)$ is called a deformation if for each local section S: $W \supset U \to E$ and each compact $K \subset U$, there is an integrable vector field X' on $J^\infty(\pi)$ such that X and X', restricted to $S_\infty(K)$ are equal, where S_∞: $U \to J^\infty(\pi)$ is the map assigning to each $u \in U$, the ∞-jet of S in u. One can think of *vertical deformation* X such that for each $s \in J^\infty(\pi)$, $d\pi_\infty(X(s)) = 0$, as vector fields on the space of (local) sections of π.

Notice finally that for each $X \in T_a(J^\infty(\pi))$, there are a total vector field H and a vertical integrable vector field $\tilde{X}$ such that $X = H(s) + \tilde{X}(s)$.

Definition (finite type) (2.7). We say that a vector field X on $J^\infty(\pi)$ is of finite type if for each $s \in J^\infty(\pi)$ there is a $k_0 \in \mathbf{N}$ such that for each $k \geqslant k_0$, there is a neighborhood U of $\pi^k_\infty(s)$ in $J^k(\pi)$ such that for each pair s', $s'' \in J^\infty(\pi)$ with $\pi^k_\infty(s') = \pi^k_\infty(s'') \in U$, $X_k(s') = X_k(s'')$. X_k is again the map from $J^\infty(\pi)$ to $T(J^k(\pi))$ as in Lemma (2.5). Notice that a total nonzero vector field is *not* of finite type, and also that if a vector field X is not of finite type, it *is the limit of a sequence of vector fields of finite type*, i.e., there is a sequence of vector fields $X^1, X^2, X^3, \ldots$ such that for each smooth $f: J^\infty(\pi) \to \mathbf{R}$ and each $s \in J^\infty(\pi)$, there are a neighborhood U of s in $J^\infty(\pi)$ and a $k \in \mathbf{N}$ such that $(X - X^k)|(f)U = 0$. It is enough to make X^i to be of finite type such that $(X - X^i)_k = 0$ whenever $k < i$. In this case we define X *to be the limit of* X^i for $i \to \infty$.

Definition (2.8). A k-form ω on $J^\infty(\pi)$ is a multilinear alternating map, assigning to each k-tuple of vector fields $X_1, \cdots, X_k$ on $J^\infty(\pi)$ a smooth function $\omega(X_1, \cdots, X_k)$ on $J^\infty(\pi)$ in such a way that $\omega(X_1, \cdots, X_k)(s)$ is completely determined by ω and $X_1(s), \cdots, X_k(s)$. We denote by $\omega(s)$ the induced alternating k-linear map from $T_s(J^\infty(\pi))$ to $\mathbf{R}$.

Lemma (2.9). *Let ω be a k-form on $J^\infty(\pi)$. Then there is a sequence of open sets $U_i \subset J^i(\pi)$ and k-forms ω_i on U_i such that*

1. $(\pi^i_\infty)^{-1}(U_i) \supset (\pi^{i-1}_\infty)^{-1}(U_{i-1})$,
2. $\bigcup_{i\in\mathbf{N}}(\pi^i_\infty)^{-1}(U_i) = J^\infty(\pi)$,
3. $(\pi^{i-1}_i)^*\omega_{i-1} = \omega_i|(\pi^{i-1}_i)^{-1}U_{i-1}$,
4. *for each $s \in (\pi^i_\infty)^{-1}U_i$*

$$\omega(s)(X_1(s), \cdots, X_k(s)) = \omega_i(\pi^i_\infty(s))(d\pi^i_\infty(X_1(s)), \cdots, d\pi^i_\infty(X_k(s))).$$

Each such sequence $\{U_i, \omega_i\}_{i\in\mathbf{N}}$ satisfying conditions 1, 2 *and* 3 *uniquely determines a k-form on $J^\infty(\pi)$; two such sequences $\{U_i, \omega_i\}$ and $\{U_i', \omega_i'\}$ determine the same k-form if and only if $\omega_i|U_i \cap U_i' = \omega_i^*|U_i \cap U_i'$ for all $i \in \mathbf{N}$.*

Proof. We choose $s \in J^\infty(\pi)$ and want to show that there are an $i \in \mathbf{N}$, a neighborhood U of $\pi^i_\infty(s)$ in $J^i(\pi)$ and a differential form $\tilde{\omega}_i$ on U such that for each $s' \in (\pi^i_\infty)^{-1}(U)$

$$\omega(s')(X_1(s'), \cdots, X_k(s')) = \tilde{\omega}_i(\pi^i_\infty(s')(d\pi^i_\infty(X_1(s')), \cdots, d\pi^i_\infty(X_k(s')))).$$

Suppose that such an i does not exist. Then there is a sequence of points $\{p_j\}_{j\in\mathbf{N}}$ in $J^\infty(\pi)$, converging to s, such that $\omega(p_j)(X_1(p_j), \cdots, X_k(p_j))$ is not determined by the projections $d\pi^j_\infty(X_1(p_j)), \cdots, d\pi^j_\infty(X_k(p_j))$. This means that, for suitable vector fields $X_1, \cdots, X_k$, the function $\omega(X_1, \cdots, X_k)$ is not constant on any of the sets $(\pi^j_\infty)^{-1}(\pi^j_\infty(p_j)), j = 1, 2, \ldots$. But this implies that $\omega(X_1, \cdots, X_k)$ is not smooth and we have the required contradiction.

Next we take for each $s \in J^\infty(\pi)$ an $i(s)$ and $U(s) \subset J^{i(s)}(\pi)$ as above, where $U(s)$ is an open neighborhood of $\pi^{i(s)}_\infty(s)$. Now $U_{i_0} = \bigcup_{\{s\in J(\pi)|i(s)\leqslant i_0\}}(\pi^{i(s)}_{i_0})^{-1}U(s)$. By the above construction there is a unique ω_i on U_i with the required properties.

The rest of the lemma is trivial.

Definition (2.10). We denote by $\mathcal{H}^k_l(\pi)$ the vector space of those $(k + l)$-forms ω on $J^\infty(\pi)$ with $\omega(X_1, \cdots, X_{k+l})$ zero if among $X_1, \cdots, X_{k+l}$ there are more than l horizontal vector fields (a vector field X is horizontal if each

$X(s)$ is horizontal), or more than k vertical vector fields (a vector field X is vertical if each $X(s)$ is vertical).

Lemma (2.11). *Each element $\omega \in \mathcal{H}_l^k(\pi)$ determines a map E_ω, which assigns to each local section S: $W \supset U \to E$ and k-tuple $\bar{X}_1, \cdots, \bar{X}_k$ of vertical symmetries of π an l-form on U, which is defined by $E_\omega(S; \bar{X}_1, \cdots, \bar{X}_k) = (S_\infty)^*(X_1, \cdots, X_k)$ where $X_1, \cdots, X_k$ are the vertical integrable vector fields corresponding to $\bar{X}_1, \cdots, \bar{X}_k$, and $S_\infty(u)$ is the ∞-jet of S in u.*

This map E_ω satisfies:

1. $E_\omega(S; \bar{X}_1, \cdots, \bar{X}_k) = E_\omega(S; \bar{X}'_1, \cdots, \bar{X}'_k)$ if $\bar{X}_i|\text{Im}(S) = \bar{X}'_i|\text{Im}(S)$,
2. E_ω is alternating and multilinear (over $\mathbf{R}$) in $\bar{X}_1, \cdots, \bar{X}_k$,
3. for each $s \in J^\infty(\pi)$ there are some $i \in \mathbf{N}$ and neighborhood U of $\pi^i_\infty(s)$ in $J^i(\pi)$ such that, as far as $S_\infty(u) \in (\pi^i_\infty)^{-1}U$, $(E_\omega(S; \bar{X}_1, \cdots, \bar{X}_k))u$ depends in a smooth way on (and is determined by) the i-jet of S in u and the i-jets of $\bar{X}_i|\text{Im}(S), \cdots, \bar{X}_k|\text{Im}(S)$ in $S(u)$.

Also, if E is a map which assigns to each local section S: $W \supset U \to E$ and each k-tuple $\bar{X}_1, \cdots, \bar{X}_k$ of vertical symmetries of π, an l-form on U in such a way that conditions 1, 2 and 3 above are satisfied, then there is a unique $\omega \in \mathcal{H}_l^k(\pi)$ such that $E = E_\omega$.

Proof. Trivial.

Remark (2.12). The operations Λ, ι_X can be defined in the usual way. Also the d-operator could be defined now, but we shall pospone this until after the discussion of the Lie derivatives.

3. Lie-derivative and exterior derivatives

Usually the definition of Lie derivative is based on the time t integral of vector fields (locally and for small t). On $J^\infty(\pi)$, integral curves of vector fields do not always exist, and if they do they are not always unique. On the other hand, if a vector field X on $J^\infty(\pi)$ is integrable, and $\bar{X}$ is the corresponding symmetry of π, then the time t integral $\mathcal{D}_{X,t}$ maps $s \in J^\infty(\pi)$ to the ∞-jet of $\mathcal{D}_{\bar{X},t} \circ S \circ \mathcal{D}_{\pi(\bar{X}),-t}$ at $\mathcal{D}_{\pi(\bar{X}),t}(\pi_\infty(s))$ provided that S is a local section representing s.

Definition (3.1). If X is a vector field on $J^\infty(\pi)$, and f a smooth function on $J^\infty(\pi)$, then we define the Lie derivative $L_X f$ of f with respect to X as $L_X f = X(f)$.

If X_1, X_2 are two vector fields on $J^\infty(\pi)$, then we define the Lie derivative $L_{X_1}X_2$ by

$$L_{X_1}(X_2(f)) = (L_{X_1}X_2)(f) + X_2(L_{X_1}(f))$$

or

$$(L_{X_1}X_2)(f) = X_1(X_2(f)) - X_2(X_1(f)).$$

It is easy to see that $L_{X_1}X_2$ thus defined is again a vector field. $L_{X_1}x_2$ is also denoted by $[X_1, X_2]$.

If X is a vector field, and ω a k-form on $J^\infty(\pi)$, then the Lie derivative $L_X\omega$ is determined by

$$L_X(\omega(X_1, \cdots, X_k)) = (L_X\omega)(X_1, \cdots, X_k) + \sum_{i=1}^{k} \omega(X_1, \cdots, L_XX_i, \cdots, X_k),$$

or

$$(L_X\omega)(X_1, \cdots, X_k) = X(\omega(X_1, \cdots, X_k)) - \sum_{i=1}^{k} \omega(X_1, \cdots, [X, X_i], \cdots, X_k).$$

Remark (3.2). In case X is an integrable vector field on $J^\infty(\pi)$, the above definitions are equivalent with

$$L_Xf = \lim_{h\to 0} \frac{1}{h}(f \circ \mathfrak{D}_{X,h} - f),$$

$$L_XX' = \lim_{h\to 0} \frac{1}{h}\left(X' - (\mathfrak{D}_{X,h})_*X'\right),$$

$$L_X\omega = \lim_{h\to 0} \frac{1}{h}(\mathfrak{D}^*_{X,h}\omega - \omega),$$

where $(\mathfrak{D}_{X,h})_*$ denotes the induced map for vector fields $((\mathfrak{D}_{X,h})X')(f) = (X'(f \circ \mathfrak{D}_{X,h})) \circ \mathfrak{D}_{X,-h}$, and $\mathfrak{D}^*_{X,h}$ denotes the induced map for differential forms, defined as in the finite dimensional case.

Theorem (3.3). *Let X_1, X_2 be vector fields on $J^\infty(\pi)$.*

1. *If X_1, X_2 are integrable with corresponding symmetries $\overline{X}_1, \overline{X}_2$ of π on E, then $[X_1, X_2]$ is integrable with corresponding symmetry $[\overline{X}_1, \overline{X}_2]$; for the projections we have $[\pi(X_1), \pi X_2] = \pi[X_1, X_2]$.*

2. *If X_1 is integrable with symmetry $\overline{X}_1$ and projection $\pi(\overline{X}_1)$, and X_2 is the total vector field of $\overline{X}_2$, then $[X_1, X_2]$ is the total vector field of $[\pi(\overline{X}_1), \overline{X}_2]$.*

3. *If X_1 and X_2 are total vector fields of $\overline{X}_1$ and $\overline{X}_2$, respectively, then $[X_1, X_2]$ is the total vector field of $[\overline{X}_1, \overline{X}_2]$.*

Proof. If X is an integrable vector field on $J^\infty(\pi)$, then $(\mathfrak{D}_{X,t})_*$, maps integrable (resp. total) vector fields to integrable (resp. total) vector fields, although it is only defined locally and for small t. From this it follows that the *Lie bracket* (is Lie derivative or Lie product) in case 1 is integrable and in case 2 is total. By evaluating this Lie product on functions of the form

$f = \tilde{f} \circ \pi_\infty^0$ and $f = \bar{f} \circ \pi_\infty$ where $\tilde{f}, \bar{f}$ are smooth functions on $E = J^0(\pi)$ and W respectively we obtain the above statements for the cases 1 and 2.

In order to deal with case 3, we consider the vector fields X_1, X_2, $\bar{X}_1$, $\bar{X}_2$, take a local section S: $W \supset U \to E$ of π, and denote again by S_∞ the corresponding map $U \to J^\infty(\pi)$. Then, for any smooth function f: $J^\infty(\pi) \to \mathbf{R}$ and $u \in U$, $(X_1(f))(S^\infty(u)) = (\bar{X}_1(f \circ S^\infty))(u)$ so that

$$\begin{aligned}([X_1, X_2]f)(S^\infty(u)) &= (X_1(X_2(f))) - X_2(X_1(f))(S^\infty(u)) \\ &= \{\bar{X}_1(X_2(f) \circ S^\infty) - \bar{X}_2(X_1(f) \circ S^\infty)\}(u) \\ &= \{\bar{X}_1(\bar{X}_2(f \circ S^\infty)) - \bar{X}_2(\bar{X}_1(f \circ S^\infty))\}(u).\end{aligned}$$

Since the last expression equals the results oftained by applying the total vector field of $[\bar{X}_1, \bar{X}_2]$ to f on u, the proof is complete.

Definition and proposition (3.4). There is a unique operator d which assigns to each form ω on $J^\infty(\pi)$ a form $d\omega$ such that the following formula holds: $L_X\omega = \iota_X\, d\omega + d\iota_X\omega$ for each vector field X and differential form ω. d is called the *exterior derivative*.

Proof (As in the finite dimensional case). If ω is a 0-form (or function), then $L_X\omega = \iota_X\, d\omega + d\iota_X\omega$; $\iota_X\omega = 0$, $L_X\omega = X(\omega)$, so $d\omega(X) = X(\omega)$. For ω a 1-form we have $L_X\omega = \iota_X\, d\omega + d\iota_X\omega$, so

$$\begin{aligned}d\omega(X_1, X_2) &= (L_{X_1}\omega)X_2 - (d(\iota_{X_1}\omega))(X_2) \\ &= X_1(\omega(X_2)) - \omega([X_1, X_2]) - X_2(\omega(X_1)).\end{aligned}$$

By induction, for a k-form ω we find

$$\begin{aligned}d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j}\omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k).\end{aligned}$$

Notice that, since our construction is made in the same way as for the finite dimensional case, if a k-form ω is given by the sequence $\{(U_i, \omega_i)\}_{i\in\mathbf{N}}$ as in Lemma (2.9), the corresponding sequence for $d\omega$ is $\{(U_i, d\omega_i)\}_{i\in\mathbf{N}}$.

It is clear that the d operator defined by the above formula is unique.

Lemma (3.5). *If* $\omega \in \mathcal{H}_l^k(\pi)$, *then* $d\omega \in \mathcal{H}_l^{k+1}(\pi) \oplus \mathcal{H}_{l+1}^k(\pi)$.

Proof. We take $s \in J^\infty(\pi)$ and a number of vectors $X_0, X_i, \ldots, X_{k+l} \in T_s(J^\infty(\pi))$ each of which is either horizontal or vertical. We have to show that if less than l or more than $l+1$ of them are horizontal, $d\omega(s)(X_0, \ldots, X_{k+l}) = 0$. For this we extend $X_0, \ldots, X_{k+1}$ to vector fields each of which is either total or vertical integrable. Now the lemma follows as a direct application of (3.3) and (3.4).

Definition (3.6). The operators ∂: $\mathcal{H}_l^k(\pi) \to \mathcal{H}_l^{k+l}(\pi)$ and D: $\mathcal{H}_l^k(\pi) \to \mathcal{H}_{l+1}^k(\pi)$ are defined by $d = \partial + D$. ∂ is called the *vertical exterior derivative*, and D the *horizontal exterior derivative*.

Remark (3.7). For $\omega \in \mathcal{H}_l^k(\pi)$ and E_ω given Lemma (2.11), $E_{D\omega}$ is determined by $E_{D\omega}(S; \overline{X}_1, \ldots, \overline{X}_k) = (-1)^k d(E_\omega(S; \overline{X}_1, \ldots, \overline{X}_k))$, where S is a local section of π, and $\overline{X}_1, \ldots, \overline{X}_k$ are vertical symmetries.

4. Exactness theorems

In this section we are concerned with the following diagram and the diagram obtained from it by replacing each $\mathcal{H}_l^k(\pi)$ by its sheaf $\tilde{\mathcal{H}}_l^k(\pi)$ of germs of sections (see [1]).

$$
\begin{array}{ccccccc}
\mathcal{H}_n^0(\pi) & \xrightarrow{\partial} & \mathcal{H}_n^1(\pi) & \xrightarrow{\partial} & \mathcal{H}_n^2(\pi) & \to & \cdots \\
\uparrow D & & \uparrow D & & \uparrow D & & \\
\mathcal{H}_{n-1}^0(\pi) & \xrightarrow{\partial} & \mathcal{H}_{n-1}^1(\pi) & \xrightarrow{\partial} & \mathcal{H}_{n-1}^2(\pi) & \to & \cdots \\
\vdots & & \vdots & & \vdots & & \\
\mathcal{H}_1^0(\pi) & \xrightarrow{\partial} & \mathcal{H}_1^1(\pi) & \xrightarrow{\partial} & \mathcal{H}_1^2(\pi) & \to & \\
\uparrow D & & \uparrow D & & \uparrow D & & \\
\mathcal{H}_0^0(\pi) & \xrightarrow{\partial} & \mathcal{H}_0^1(\pi) & \xrightarrow{\partial} & \mathcal{H}_0^2(\pi) & \to & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{R} & & 0 & & 0 & &
\end{array}
$$

We are interested in various exactness properties of these diagrams. First of all we have the Poincaré lemma.

Theorem (4.1) (*Poincaré*). *If* $\alpha_0 \in (\tilde{\mathcal{H}}_m^0(\pi))_s$, $\alpha_1 \in (\tilde{\mathcal{H}}_{m-1}^1(\pi))_s, \ldots, \alpha_m \in (\tilde{\mathcal{H}}_0^0(\pi))_s$, $m \geqslant 1$, *and* $D\alpha_0 = 0$, $\partial\alpha_0 + D\alpha_1 = 0, \ldots, \partial\alpha_{m-1} + D\alpha_m = 0$, $\partial\alpha_m = 0$, *then there are* $\beta_0 \in (\tilde{\mathcal{H}}_{m-1}^0(\pi))_s, \ldots, \beta_{m-1} \in (\tilde{\mathcal{H}}_0^{m-1}(\pi))_s$ *such that* $D\beta_0 = \alpha_0$, $\partial\beta_0 + D\beta_1 = \alpha_1, \ldots, \partial\beta_{m-2} + D\beta_{m-1} = \alpha_{m-1}$, $\partial\beta_{m-1} = \alpha_m$.

We use the conventions: $\tilde{\mathcal{H}}_l^k(\pi) = 0$ whenever $l > n$, $l < 0$ or $k < 0$, where $(\tilde{\mathcal{H}}_l^k(\pi))_s$ denotes the stalk of germs of sections at $s \in J^\infty(\pi)$ in $\tilde{\mathcal{H}}_l^k(\pi)$. We also assume $n \geqslant 1$. The main result of this section is

D-Exactness theorem (4.2). *Let* $\omega \in (\tilde{\mathcal{H}}_l^k(\pi))_s$, $0 < l < n = \dim(W)$, *and* $D\omega \in (\tilde{\mathcal{H}}_{l+1}^k(\pi))_s = 0$. *Then there is an* $\eta \in (\tilde{\mathcal{H}}_{l-1}^k(\pi))_s$ *such that* $D\eta = \omega$. D: $\mathcal{H}_0^k(\pi) \to \mathcal{H}_l^k(\pi)$ *is injective for* $k > 0$, *the kernel of* D: $\mathcal{H}_0^0(\pi) \to \mathcal{H}_l^0(\pi)$ *consists of the locally constant functions in* $\mathcal{H}_0^0(\pi)$, *and the image of* D: $\tilde{\mathcal{H}}_{n-1}^k(\pi) \to \tilde{\mathcal{H}}_n^k(\pi)$ *is characterized as follows.*

$\omega \in (\tilde{\mathcal{H}}_n^k(\pi))_s$, $k > 0$, *is in the image of* D *if and only if there are a* *representative $\hat{\omega}$ of ω in $\mathcal{H}_n^k(\pi)$, and a neighborhood U of s in $J^\infty(\pi)$ such that for each local section S, defined on a neighborhood of $\pi_\infty(s)$, and each $V \subset \bar{V} \subset (S_\infty)^{-1}(U)$, V a bounded oriented open subset of W, $\int_V E_{\hat{\omega}}(S; \bar{X}_1, \ldots, \bar{X}_k)$ depends only on the germs of the vertical symmetries $\bar{X}_1, \ldots, \bar{X}_k$ along $S(\partial V)$ (and so S, V, $\hat{\omega}$ but not on $\bar{X}_1, \ldots, \bar{X}_k$ away from $S(\partial V)$).*

Remark (4.3). The proof of the above theorem is somewhat complicated and therefore postponed until the end of this section. We first derive a number of consequences from it.

First we note that our diagram is anticommutative: because $d \circ d = 0$, $d = D + \partial$ we have $\partial \circ \partial = 0$, $D \circ D = 0$ and $\partial \circ D + D \circ \partial = 0$.

∂-Exactness theorem (4.4). *$\tilde{\mathcal{G}}^i(\pi)$ denotes the quotient sheaf $\tilde{\mathcal{H}}_n^i(\pi)/D(\tilde{\mathcal{H}}_{n-1}^i(\pi))$. If $\omega \in (\tilde{\mathcal{G}}^i(\pi))_s$, $i > 0$, with $\partial\omega = 0$ (∂ is defined here because $\partial(D(\tilde{\mathcal{H}}_{n-1}^i(\pi))) \subset D(\tilde{\mathcal{H}}_{n-1}^{i+1}(\pi))$), then there is an $\eta \in \tilde{\mathcal{G}}^{i-1}(\pi)$ such that $\partial\eta = \omega$, and ∂: $\tilde{\mathcal{G}}^0(\pi) \to \tilde{\mathcal{G}}^1(\pi)$ is injective.*

Proof. First we show that ∂: $\tilde{\mathcal{G}}^0(\pi) \to \tilde{\mathcal{G}}^1(\pi)$ is injective. Suppose not; then there is an $\alpha \in (\tilde{\mathcal{G}}^0(\pi))_s$ with $\partial\alpha = 0$. By definition of $\tilde{\mathcal{G}}^0(\pi)$ there is $\beta_0 \in (\tilde{\mathcal{H}}_n(\pi))_s$ with $[\beta_0] = \alpha$ and hence $\partial\beta_0 \in \mathrm{Im}(D$: $\tilde{\mathcal{H}}_{n-1}^1(\pi) \to \tilde{\mathcal{H}}_n^1(\pi))$. Choose $\beta_1 \in (\tilde{\mathcal{H}}_n^1(\pi))_s$ such that $\partial\beta_0 + D\beta_1 = 0$. With induction we go on and find $\beta_2, \ldots, \beta_n$ with $\beta_i \in (\tilde{\mathcal{H}}_{n-i}^i(\pi))_s$ such that $\partial\beta_i + D\beta_{i+1} = 0$, $i = 0, 1, \cdots, n - 1$. Then $\partial\beta_n = 0$ because $D\partial\beta_n = -\partial D\beta_n = \partial\partial\beta_{n-1} = 0$, and D in this case is injective. Applying the Poincaré lemma to $(\beta_0, \ldots, \beta_n)$ one obtains $(\gamma_0, \ldots, \gamma_{n-1})$, $\gamma_i \in (\tilde{\mathcal{H}}_{n-i-1}^i(\pi))_s$, $D\gamma_0 = \beta_0$, $\partial\gamma_0 + D\gamma_1 = \beta_1, \cdots$. So $\beta_0 \in \mathrm{Im}(D)$ and hence $\alpha = 0$.

Next $\partial \circ \partial = 0$ (also for the sheaves $\tilde{\mathcal{G}}(\pi)$), so we have to prove that if $\alpha \in (\tilde{\mathcal{G}}^1(\pi))_s$ and $\partial\alpha = 0$, there is $\gamma \in (\tilde{\mathcal{G}}^0(\pi))_s$ with $\partial\gamma = \alpha$. Choose $\beta_0 \in (\tilde{\mathcal{H}}_n^1(\pi))_s$ with $[\beta_0] = \alpha$. Then $\partial\beta_0 \in \mathrm{Im}(D)$, so there is $\beta_1 \in (\tilde{\mathcal{H}}_{n-1}^2(\pi))_s$ such that $\partial\beta_0 + D\beta_1 = 0$. Now we proceed as in the above case to find $\beta_1, \ldots, \beta_n$ and then $\gamma_0, \ldots, \gamma_n$ with $\gamma_0 \in (\tilde{\mathcal{H}}_n^0(\pi))_s$, $\gamma_1 \in (\tilde{\mathcal{H}}_{n-1}^1(\pi))_s$, $\partial\gamma_0 + D\gamma_1 = \beta_0$. Hence $\partial[\gamma_0] = \alpha$, so we may take $\gamma = [\gamma_0]$.

In the same way one now proceeds easily to prove that ∂ makes $\tilde{\mathcal{G}}^i(\pi)$ an exact complex. This completes the proof.

Remark (4.5). A stronger version of (4.4) can also be proved using a kind of "fibrewise Poincaré lemma"; in this way one can show that if $\omega \in (\tilde{\mathcal{H}}_l^k(\pi))_s$ and $\partial\omega = 0$, $k > 0$, then there is an $\eta \in (\tilde{\mathcal{H}}_l^{k-1}(\pi))_s$ with $\partial\eta = \omega$; see [3.]. We shall however not need this fact here.

D-cohomology theorem (4.6). *Let $\mathcal{G}^i(\pi)$ denote the set of global sections of the sheaf $\tilde{\mathcal{G}}^i(\pi)$ defined in Theorem (4.4); the canonical map $\mathcal{H}_n^i(\pi) \to \mathcal{G}_n^i(\pi)$ is denoted by D (one has to be careful; for a correct interpretation of (4.1), D: $\mathcal{H}_n^i(\pi) \to \mathcal{H}_{n+1}^i(\pi)$ is the zero map!). Then*

$$\frac{\operatorname{Ker}(D\colon \mathcal{H}_k^i(\pi) \to \mathcal{H}_{k+1}^i(\pi))}{\operatorname{Im}(D\colon \mathcal{H}_{k-1}^i(\pi) \to \mathcal{H}_k^i(\pi))} \simeq \begin{cases} 0, & \text{if } i \neq 0,\ 0 < k < n, \\ H^k(E; \mathbf{R}), & \text{if } i = 0,\ 0 < k < n; \end{cases}$$

$D\colon \mathcal{H}_0^i(\pi) \to \mathcal{H}_1^i(\pi)$ is injective provided $i > 0$, and

$$\frac{\operatorname{Ker}(D\colon \mathcal{H}_n^i(\pi) \to \mathcal{G}^i(\pi))}{\operatorname{Im}(D\colon \mathcal{H}_{n-1}^i(\pi) \to \mathcal{H}_n^i(\pi))} \simeq \begin{cases} 0, & \text{if } i \neq 0, \\ H^n(E; \mathbf{R}), & \text{if } i = 0. \end{cases}$$

Proof. By (2.1), $\tilde{\mathcal{H}}_j^i(\pi)$ is a fine sheaf for all i, j; see [1]. Hence

$$0 \to \mathbf{R} \to \tilde{\mathcal{H}}_0^0(\pi) \xrightarrow{D} \tilde{\mathcal{H}}_1^0(\pi) \xrightarrow{D} \cdots \xrightarrow{D} \tilde{\mathcal{H}}_n^0(\pi)$$

is a fine resolution of the constant sheaf $\mathbf{R}$, and applying the argument used in [1] to prove the Rham's theorem, one finds that for $0 \leqslant k < n$

$$\operatorname{Ker}(D\colon \mathcal{H}_k^0(\pi)) \to \mathcal{H}_{k+1}^0(\pi)/\operatorname{Im}(D\colon \mathcal{H}_{k-1}^0(\pi) \to \mathcal{H}_k^0(\pi)) \simeq H^k(E;\mathrm{r}).$$

This argument still works if the last sheaf of the resolution is not fine; hence we can add $\xrightarrow{D} \tilde{\mathcal{G}}_n^0(\pi) \to 0$, and for the case $i = 0$ the theorem follows.

For $i > 0$, one has to deal with a "fine" resolution of the zero sheaf, and hence we obtain the exactness.

Remark (4.7). Using the same arguments one can show that $D\colon \mathcal{H}_n^i(\pi) \to \mathcal{G}^i$ is surjective for $i > 0$.

G-cohomology theorem (4.8). *For $i > 0$,*

$$\operatorname{Ker}(\partial\colon \mathcal{G}^i(\pi) \to \mathcal{G}^{i+1}(\pi))/\operatorname{Im}(D\partial\colon \mathcal{H}_n^{i-1}(\pi) \to \mathcal{G}^i(\pi)) \simeq H^{n+i}(E\colon \mathbf{R}).$$

Proof. We shall use the following procedure.

First we show how to associate to each $\alpha \in \mathcal{G}^i(\pi)$ with $\partial\alpha = 0$ an $(n+i)$ form β on $J^\infty(\pi)$ with $d\beta = 0$. The construction of such β is not unique but we prove that each closed $(n+i)$-form $\tilde{\beta}$ on $J^\infty(\pi)$ can be obtained by the above mentioned construction, from some $\tilde{\alpha} \in \mathcal{G}^i(\pi)$ with $\partial\tilde{\alpha} = 0$ and that

$\alpha \in \operatorname{Im}(D\partial)$ if and only if β is in the image of d (independent of the choices). From this one deduces immediately that $\operatorname{Ker}(\partial\colon \mathcal{G}^i(\pi) \to \mathcal{G}^{i+1}(\pi))/\operatorname{Im}(D\partial\colon \mathcal{H}_n^{i-1}(\pi) \to \mathcal{G}^i(\pi))$ is isomorphic with the $(n+i)$th de Rham cohomology group of $J^\infty(\pi)$, which course equals $H^{n+i}(E; \mathbf{R})$.

Now we come to the construction of β for given $\alpha \in \mathcal{G}^i(\pi)$ with $\partial\alpha = 0$. By (4.7) there is a $\beta_0 \in \mathcal{H}_n^i(\pi)$ such that $D\beta_0 = \alpha$. Since $\partial\alpha = 0$, we have $-\partial D\beta_0 = D\partial\beta_0 = 0$, and hence by (4.2) there is a $\beta_1 \in \mathcal{H}_{n-1}^{i+1}(\pi)$ such that $D\beta_1 + \partial\beta_0 = 0$. By induction, one finds now $\beta_2 \in \mathcal{H}_{n-2}^{i+2}(\pi), \ldots, \beta_n \in \mathcal{H}_n^{i+n}(\pi)$ such that $D\beta_2 + \partial\beta_1 = 0, \cdots, D\beta_n + \partial\beta_{n-1} = 0$. Then $D\partial\beta_n = -\partial D\beta_n = \partial \circ \partial\beta_{n-1} = 0$ and hence $\partial\beta_n = 0$ by (4.2). Now we define $\partial = \sum_{i=0}^n \beta_i$; clearly $d\beta = 0$. For any $(n+i)$-form $\tilde{\beta}$ (with $d\tilde{\beta} = 0$) one has $\tilde{\beta} = \sum_{j=0}^n \tilde{\beta}_j$ with $\tilde{\beta}_j \in \mathcal{H}_{n-j}^{i+j}(\pi)$ and $\partial\tilde{\beta}_j + D\tilde{\beta}_{j+1} = 0$. Hence, if $\tilde{\alpha} = D\tilde{\beta}_0$, then $\tilde{\beta}$ could be obtained from $\tilde{\alpha}$ by the above construction.

Finally, assume there is a $\gamma_0 \in \mathcal{K}_n^{i-1}(\pi)$ with $D\partial\gamma_0 = \alpha$: $\beta_0, \ldots, \beta_n$ are still as above. Then $D(\partial\gamma_0 - \beta_0) = 0$, and hence there is a $\gamma_1 \in \mathcal{K}_{n-1}^{i}(\pi)$ such that $\beta_0 = \partial\gamma_0 + D\gamma_1$. Similarly, one makes $\gamma_2 \in \mathcal{K}_{n-2}^{i+1}(\pi)$ with $D\gamma_2 + \partial\gamma_1 = \beta_1$, etc. Then $d\gamma = \beta$ if $\gamma = \sum_{i=0}^{n} \gamma_i$. This completes the proof.

Corollary (4.9). *If* $i \geqslant 2$, *then*

$$\operatorname{Ker}(\partial\colon \mathcal{G}^i(\pi) \to \mathcal{G}^{i+1}(\pi))/\operatorname{Im}(\partial\colon \mathcal{G}^{i-1}(\pi) \to \mathcal{G}^i(\pi)) \simeq H^{n+i}(E).$$

Proof of Theorem (4.2). For this we have to consider *form-operators*:

A p-form-operator of order k on $\mathbf{R}^n$ is a linear map δ assigning to each smooth function $f\colon \mathbf{R}^n \to \mathbf{R}$ a p-form $\delta(f)$ on $\mathbf{R}^n$ such that $\delta(f)(x)$ depends (smoothly) only on the k-jet of f in x. Clearly, if δ is a p-form-operator of order k on $\mathbf{R}^n$, then $(f \to d\delta(f))$ is a $(p+1)$-form-operator of order $k+1$ on $\mathbf{R}^n$.

Definition (4.10). A p-form-operator δ on $\mathbf{R}^n$ is said to be *closed* if:

for $p < n$, $d\delta(f) = 0$ for all smooth f,

for $p = n$, for each bounded oriented $A \subset \mathbf{R}^n$, $\int_A \delta(f)$ depends only on the germ of f along ∂A.

Note that the $(p+1)$-form-operator $d\delta$ is always closed.

Remark (4.11). We shall use the following conventions:

$$\Lambda_{n,k} = \{(j_1, \cdots, j_l) | 1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_l \leqslant n, 0 \leqslant l \leqslant k\};$$

for $J = (j_1, \cdots, j_l) \in \Lambda_{n,l}$, $\partial_J f = \partial_{j_1, \ldots, j_l} f = \partial^l f/\partial x_{j_1}, \ldots, {}_{j_l}\partial_{x_{j_l}}$.

A p-form operator δ of order k on $\mathbf{R}^n$ can always be written in a unique way as

$$\delta(f) = \sum_{\substack{I \in \Lambda_{n,k} \\ 1 < j_1 < \cdots < j_p < n}} \delta^I_{j_1, \ldots, j_p}(x) \cdot (\partial_I f)(x) \cdot dx_{j_1} \wedge \cdots \wedge dx_{j_p},$$

The h-jet of such δ in x is supposed to be determined by the h-jets of the functions $\delta^I_{j_1, \ldots, j_p}$ in x.

Lemma (4.12). *There is a linear map P which assigns to each closed p-form-operator δ of order k on $\mathbf{R}^n$ a $(p-1)$-form-operator $P(\delta)$ of order $(k-1)$ on $\mathbf{R}^n$ such that $d(P(\delta))(f) = \delta(f)$ for each smooth function $f\colon \mathbf{R}^n \to \mathbf{R}$, and that*

$(P(\delta))(f)(x)$ is determined by and depends smoothly on the $(k-1)$-jet of f in x and the h-jet of δ in x where $h \in \mathbf{N}$ is some integer determined by p, k and n.

Proof. If $p = 0$, $\delta(f)$ has to be a constant function for each f. But $(\delta(f))(x)$ depends only on the k-jet of f in x (in a linear way), hence $\delta(f) = 0$ for each f. So $P(\delta) = 0$ if $p = 0$.

If $p = n$, we may write $\delta(f) = \sum_{I \in \Lambda_{n,k}} \delta^I(x) \cdot \partial_I f(x) \cdot dx_1 \wedge \cdots \wedge dx_n$.

Then

$$\delta(f) = \sum_{(j_1,\cdots,j_l)\in\Lambda_{n,k}} -\left\{\frac{\partial\delta^{(j_1,\cdots,j_l)}}{\partial x_{j_l}}(x)\cdot\partial_{j_1,\cdots,j_{l-1}}f(x)\cdot dx_1\wedge\cdots\wedge dx_n\right\}$$

$$+d\Bigg(\sum_{(j_1,\cdots,j_l)\in\Lambda_{n,k}}(-1)^{j_l+1}\cdot\delta^{(j_1,\cdots,j_l)}(x)\cdot\partial_{j_1,\cdots,j_{l-1}}f(x)$$

$$\cdot dx_1\wedge\cdots\wedge d\hat{x}_{j_l}\wedge\cdots\wedge dx_n\Bigg)$$

$$= \delta_1(f) + d\delta_2(f),$$

with δ_1 and δ_2 as $n-$ and $(n-1)$-form-operators of order $(k-1)$ respectively. δ_1 is of course again closed, so we can aply the above construction again on δ_1, and repeat until we have $\bar{\delta} = \bar{\delta}_1 + d\bar{\delta}_2$ with $\bar{\delta}_1$ and $\bar{\delta}_2$ as $n-$ and $(n-1)$-form-operators of orders 0 and $(k-1)$ on $\mathbf{R}^n$, respectively. By assumption δ is closed and hence $\bar{\delta}_1$. But it is easy to see that any closed n-form-operator of order 0 on $\mathbf{R}^n$ is identically 0, so $\delta = d\bar{\delta}_2$, and we define $P(\delta)$ in this case as $\bar{\delta}_2$ constructed in the above way. Notice that this definition of $P(\delta)$ even depends on the order of the coordinates; it is highly not unique.

If $0 < p < n$, we may write

$$\delta(f) = \sum_{\substack{I\in\Lambda_{n,k}\\ i_1<\cdots<i_p}} \delta^I_{i_1,\cdots,i_p}(x)\cdot\partial_I f(x)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

For each summand with $n\in I$ and $n = i_p$ we perform the following (with $I = (j_1,\cdots,j_l)$):

$$\begin{aligned}
\delta^I_{i_1,\cdots,i_p}&(x)\cdot\partial_I f(x)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&= \partial_n\big(\delta^I_{i_1,\cdots,i_p}(x)\cdot\partial_{j_1,\cdots,j_{l-1}}f(x)\big)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&\quad-\big(\partial_n\delta^I_{i_1,\cdots,i_p}(x)\big)\cdot(\partial_{j_1,\cdots,j_{l-1}}f)(x)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&\quad-\big(\partial_n\delta^I_{i_1,\cdots,i_p}\big)(x)\cdot(\partial_{j_1,\cdots,j_{l-1}}f)(x)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_p}\\
&\quad+(-1)^p d\big(\delta^I_{i_1,\cdots,i_p}(x)\big)\cdot(\partial_{j_1,\cdots,j_{l-1}}f)(x)\cdot dx_{i_1}\wedge\cdots\wedge dx_{i_{p-1}}\\
&\quad+(-1)^{p+1}\sum_{k\notin\{i_1,\cdots,i_p\}}\partial_k\big(\delta^I_{i_1,\cdots,i_p}(x)\cdot(\partial_{j_1,\cdots,j_{l-1}}f)(x)\big)\\
&\qquad\qquad\cdot dx_k\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{p-1}}.
\end{aligned}$$

Applying this to each summand above, one finds that $\delta(f) = \delta_1(f) + d\delta_2(f)$ with δ_2 of order $\leqslant k-1$ and δ_1 of order k but such that the summands with

dx_n only contain derivatives of f of order $\leqslant k-1$, with respect to $x_n \cdot \delta_1$ and δ_2 depend linearly on δ. Now we apply the same procedure over and over to δ_1, and obtain $\delta = \bar{\delta}_1 + d\bar{\delta}_2$ with $\bar{\delta}_2$ of order $\leqslant k-1$ and $\bar{\delta}_1$ of order $\leqslant k$ but such that in the summands with dx_n no differentiation of f with respect to x_n occurs, i.e.,

$$\bar{\delta}_1(f) = \sum_{\substack{I \in \Lambda_{n,k} \\ i_1 < \cdots < i_p}} \bar{\delta}^I_{1_{i_1,\cdots,i_p}} \partial_I f(x) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad \text{with } \bar{\delta}^I_{1_{i_1,\cdots,i_p}} = 0,$$

whenever $n \in I$ and $i_p = n$. Clearly $d\bar{\delta}_1 = 0$; we shall use this fact to prove also that $\bar{\delta}^i_{1_{i_1,\cdots,i_p}} = 0$ whenever $i_p < n$. The summand of $d\bar{\delta}_1$ with $dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_n$, $i_1 << \cdots < i_p < n$, equals

$$(-1)^p \partial_n \Big(\sum_{I \in \Lambda_{n,k}} \bar{\delta}^I_{1_{i_1,\cdots,i_p}}(x) \cdot (\partial_I f)(x) \Big) \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_n$$

$$+ \sum_{\substack{h=1,\cdots,p \\ I \in \Lambda_{n,k}}} (-1)^{h+1} \partial_{i_h} \Big(\bar{\delta}^I_{1_{i_1,\cdots,\hat{i}_h,\cdots,i_p,n}}(x) \cdot \partial_I f(x) \Big) \, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_n$$

$$= \sum_{J \in \Lambda_{n,k+1}} L^J(x) \cdot \partial_J f(x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_n,$$

where L^J is defined by the last equality.

For each $J \in \Lambda_{n,k+1}$, L^J has to be zero. For $J = (j_1, \cdots, j_{l-1}, n)$, $j_{l-1} < n$, this means that

$$L^J = (-1)^p \Big\{ \partial_n \bar{\delta}^{j_1,\cdots,j_{l-n}}_{1_{i_1,\cdots,i_p}} + \bar{\delta}^{j_1,\cdots,j_{l-1}}_{1_{i_1,\cdots,i_p}} \Big\} = 0;$$

for $J = (j_1, \ldots, j_{l-2}, n, n), j_{l-2} < n$, this means that

$$(-1)^p \Big\{ \partial_n \bar{\delta}^{j_1,\cdots,j_{l-2},n,n}_{1_{i_1,\cdots,i_p}} + \bar{\delta}^{j_1\cdots j_{l-1},n}_{1_{i_1,\cdots,i_p}} \Big\} = 0;$$

etc. Thus

$$\bar{\delta}^{j_1,\cdots,j_{l-1}}_{1_{i_1,\cdots,i_p}} = -\partial_n \bar{\delta}^{j_1,\cdots,j_{l-1},n}_{1_{i_1,\cdots,i_p}} = +\partial_{nn} \bar{\delta}^{j_{1-n},n,n}_{1_{i_1,\cdots,i_p}} = \cdots = 0$$

(we get zero when the number of superscripts exceeds k);

$$\bar{\delta}^{j_1,\cdots,j_{l-2},n}_{1_{i_1,\cdots,i_p}} = -\partial_n \bar{\delta}^{j_1,\cdots,j_{l-2},n,n}_{1_{i_1\cdots i_p}} = \partial_{nn} \bar{\delta}^{j_1,\cdots,j_{l-2},n,n,n}_{1_{i_1,\cdots,i_p}} = \cdots = 0;$$

etc. Hence we have

$$\bar{\delta}_1(f) = \sum_{\substack{J \in \Lambda_{n-1,k} \\ 0 < i_1 < i_2 < \cdots < i_{p-1} < n}} \bar{\delta}^J_{1_{i_1,\cdots,i_{p-1},n}}(x) \cdot \partial_J f(x) \cdot dx_{i_1} \wedge / \wedge dx_{i_{p-1}} \wedge dx_n.$$

Since this is a 1-parameter (namely x_n) family of $(p-1)$-form-operators of degree k on $\mathbf{R}^{n-1}$, this means that our lemma can be obtained by induction:

it holds for 0-form-operators of degree k on $\mathbf{R}^{n-p}$. If it holds for $(p-1)$-form-operators on $\mathbf{R}^{n-1}$, then it holds for p-form-operators on $\mathbf{R}^n$, since a closed p-form-operator δ can be written as $\delta = \bar{\delta}_1 + d\bar{\delta}_2$, where $\bar{\delta}_1, \bar{\delta}_2$ depend nicely on δ, and $\bar{\delta}_1$ is as constructed above, and since there is a smooth 1-parameter family of closed $(p-1)$-form-operators $\bar{\delta}_{1,x_n}$ on $\mathbf{R}^{n-1}$ such that $\bar{\delta}_1(f)(x_1, \cdots, x_n) = \bar{\delta}_{1,x_n}(f_{x_n})(x_1, \cdots, x_{n-1}) \wedge dx_n$, where $f_{x_n}: \mathbf{R}^{n-1} \to \mathbf{R}$ is defined by $f_{x_n}(x_1, \cdots, x_{n-1}) = f(x_1, \cdots, x_n)$.

One can now define $P(\delta)$ by

$$\big((P(\delta)(f))(x_1, \cdots, x_n)$$
$$= \big(P\big(\bar{\delta}_{1,x_n}\big)(f_{x_n})\big)(x_1, \cdots, x_{n-1}) \wedge dx_n + \big(\bar{\delta}_2(f)\big)(x_1, \cdots, x_n),$$

which completes the proof of the lemma.

Remark (4.13). The above lemma of course also applies to operators which assign a k-form on $\mathbf{R}^n$ to each smooth map $f: \mathbf{R}^n \to V$, where V is a fixed finite dimensional vector space. We need even more:

We say that δ is a *p-multi-form-operator of order k and multiplicity l* if δ assigns a p-form on $\mathbf{R}^n$ to each l-tuple of functions $f_1, \cdots, f_l: \mathbf{R}^n \to V$ such that $\delta(f_1, \cdots, f_l)(x)$ depends smoothly on and is determined by x and the k-jets of $f_1, \cdots, f_l$ in x and such that $\delta(f_1, \cdots, f_l)$ is multilinear and antisymmetric in $f_1, \cdots, f_l$. Also for these multi-form-operators the analogue of (4.12) holds: such δ is closed if for each $f_2, \cdots, f_l$, the operator $f_1 \to \delta(f_1, f_2, \cdots, f_l)$ is closed. Applying Lemma (4.12) we find that for each $f_2, \cdots, f_l$ there is a form-operator $P(\delta(-, f_2, \cdots, f_l))$, which is linear in $\delta(-, f_2, \cdots, f_l)$ and hence in $f_2, \cdots, f_l$, satisfies $d(P(\delta(f_1, \cdots, f_l))) = \delta(f_1, f_2, \cdots, f_l)$ for each f_1. Now define

$$(\tilde{P}(\delta))(f_1, \cdots, f_l) = \frac{1}{l!} \sum_{\sigma \in S_l} (-1)^{|\sigma|}(P(\delta(f_{\sigma(2)}, \cdots, f_{\sigma(l)})))(f_{\sigma(1)}).$$

Then we have $d(\tilde{P}(\delta)) = \delta$ for each closed multi-form-operator.

Proof of (4.2). Let $x_1, \cdots, x_n$ be a local coordinate system on a neighborhood of $\pi_\infty(s)$ in W. We also use x_i to denote $x_i \circ \pi_0$ which we shall use as coordinate function on E in order to get a complete coordinate system on a neighborhood of $\pi_\infty^0(s)$ in E we add the coordinates $y_1, \cdots, y_m$. If $\omega \in \mathcal{H}_l^k(\pi)$ then E_ω, restricted to the coordinate neighborhood, can be considered as a map, assigning to each local section S (as far as its image is in the coordinate neighborhood) an l-multi-form-operator of multiplicity k in the following way:

Let $f_1, \cdots, f_k$: $\mathbf{R}^n \to \mathbf{R}^m$ be k smooth functions: $f_i(x) = (f_i^l(x), \cdots, f_i^m(x))$. One associates to each f_i a vertical integrable vector field

X_1 along $\operatorname{Im} S_\infty$ such that $X_i(S(x)) = \Sigma_{j=1}^m f_i^j(x)\partial/\partial y_j$. $E_\omega(S; X_1, \cdots, X_k)$ is the corresponding l-form.

If the germ of ω is as in the assumptions of (4.2) then, restricted to a sufficiently small neighborhood of s, the corresponding multi-form-operators $E_\omega(S; -, \cdots, -)$ are closed, hence $\tilde{P}$ of these operators is defined. So we can define η (locally) by $(-1)^k\tilde{P}(E_\omega(S; -, \cdots, -) = E_\eta(S; -, \cdots, -))$. From the various constructions and definitions it follows that η is locally well-defined and has the required properties.

5. The main results

Definition (5.1). A variational problem or Lagrangian on π is an element $\mathcal{L} \in \mathcal{H}_n^0(\pi)$.

Indeed such $\mathcal{L}$ assigns to each (local) section S and n-form $E_\mathcal{L}(S)$ on W (see the introduction).

Lemma (5.2). *Let $A \subset W$ be a bounded open oriented subset of W, and $K \subset A$ some compact subset. Then, for each section S_t of π, defined on A and depending smoothly on t such that $S_t(w) = S_0(w)$ whenever $w \in A \setminus K$, and for each Lagrangian $\mathcal{L}$,*

$$\frac{d}{dt}\int_A E_\mathcal{L}(S_t)\bigg|_{t=0} = \int_A E_{d\mathcal{L}}(S; \bar{X}),$$

where $\bar{X}$ is a vertical symmetry of π such that for each $w \in A$, $\bar{X}(S(w)) = d/dtS_t(w)$.

Proof. Clearly $(d/dt)\int_A E_\mathcal{L}(S_t)|_{t=0} = \int_A E_{L_X\mathcal{L}}(S_0)$ where X is the vertical integrable vector field corresponding to $\bar{X}$. $L_X\mathcal{L} = d\mathcal{L}(x,-, \cdots, -)$ implies the lemma.

Remark (5.3). Lemma (5.2) remains of course true if we add to $d\mathcal{L}$ an element which belongs to $\operatorname{Im}(D\colon \mathcal{H}_{n-1}^1(\pi) \to \mathcal{H}_n^1(\pi))$. We shal use this freedom to associate to each Lagrangian $\mathcal{L}$ a particularly nice form $\Pi\, d\mathcal{L} \in \mathcal{H}_n^1(\pi)$ such that $d\mathcal{L} - \Pi\, d\mathcal{L} \in \operatorname{Im}(D)$. These forms $\Pi\, d\mathcal{L}$ will be called source forms, to be defined below.

Definition (5.4). A form $\mathcal{S} \in \mathcal{H}_n^1(\pi)$ is called a *source form*, or *source equaton*, if for each section S, $x \in \operatorname{Domain}(S)$, and vertical symmetry $\bar{X}$, $E_\mathcal{S}(S; \bar{X})(x)$ depends only on $\mathcal{S}$, S, x and $\bar{X}(S(x))$ but not on the higher derivatives of $\bar{X}$ in $S(x)$.

Lemma (5.5). *Each $\omega \in \mathcal{H}_n^1(\pi)$ can be written uniquely as $\omega = \omega_1 + \omega_2$ with ω_1 a source form and $\omega_2 \in \operatorname{Im}(D)$.*

Proof. It is clear that if $\omega \in \operatorname{Im}(D)$ and ω is a source form, then $\omega = 0$; this also holds for $\omega|U$, U an open subset of $J^\infty(\pi)$. Hence it is enough to make, for given ω, locally a source form ω_1 such that $\omega - \omega_1 \in \operatorname{Im}(D)$ (locally). In order to perform the local construction, we take local coordinates: $x_1, \cdots, x_n$ on W and $y_1, \cdots, y_m$ on E such that $x_1, \cdots, x_n, y_1, \cdots, y_m$ form a set of local coordinates on E (if we identify $x_i\pi_0$ with x_i). In these coordinates vertical integrable vector fields have the form $\overline{X} = \sum_{i=1}^m \overline{X}_i(x, y)\partial/\partial y_i$, and

$$E_\omega(S; \overline{X})(x) = \sum_{\substack{i=1 \\ 1<j_1<\cdots<j_k<n}}^{\eta} (\Omega(S))_{i;j_1,\cdots,j_k}(x) \cdot \frac{\partial^k \overline{X}_i(x, S(x))}{\partial x_{j_1}, \cdots, \partial x_{j_k}} \cdot dx_1 \wedge \cdots \wedge dx_n,$$

where $*$ means that the summation is locally finite. Now we define ω_1 locally by

$$E_{\omega_1}(S; \overline{X})(x) = \sum_{\substack{i=1 \\ 1<j_1<\cdots<j_k<n}} (-1)^k \frac{\partial k}{\partial x_{j_1} \cdots \partial x_{j_k}} (\Omega(S))_{i;j_1,\cdots,j_k}(x) \cdot \overline{X}_i(x, S(x)) \cdot dx_1 \cdot \wedge \cdots \wedge dx_n.$$

It is not difficult to see that this ω_1 has the required properties.

Definition (5.6). For $\omega \in \mathcal{H}_n^1(\pi)$, $\Pi\, \omega \in \mathcal{H}_n^1(\pi)$ is the unique source form with $\omega - \Pi\, \omega \in \operatorname{Im}(D)$. Note that there is a 1-1 correspondence between source forms and elements of $\mathcal{G}^1(\pi)$.

Remark (5.7). If $\mathfrak{L}$ is a Lagrangian, its Euler equation is $\Pi\, d\mathfrak{L}$. This follows from Lemma (5.2) and Definition (5.6). A source equation $\mathfrak{S}$ is locally variational if $D \circ \partial E = 0$; namely, this implies by the ∂ exactness theorem (4.4) (see also Tonti [4]) that for each $s \in J^\infty(\pi)$ there are a neighborhood U_s of s in $J^\infty(\pi)$ and a Lagrangian $\mathfrak{L}$, defined on U_s such that $\Pi\, d\mathfrak{L} = \mathfrak{S}\,|\,U_s$. $\mathfrak{S}$ is globally variational if there is a globally defined Lagrangian $\mathfrak{L}$ with $\mathfrak{S} = \Pi\, d\mathfrak{L}$. From the $\mathcal{G}$-cohomology theorem (4.8) we have

Main theorem (5.8). *Each locally variational source equation is globally variational provided* $H^{n+1}(E; \mathbf{R}) = 0$. *More precisely, the vector space of locally variational source equations modulo globally variational source equations is canonically isomorphic with* $H^{n+1}(E; \mathbf{R})$.

Added in proof. The main results of this paper were obtained independently by A. M. Vinogradov, Sov. Math. Dokl. **18** (1977) 1200–1204, **19** (1978) 144–148.

References

[1] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, Berlin, 1966.

[2] S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.

[3] F. Takens, *Symmetries, conversation laws and variational principles*, Lecture Notes in Math. Vol. 597, Springer, Berlin, 1977.

[4] E. Tonti, *Variational formulation of nonlinear differential equations* (I), Acad. Roy. Belg. C. (V) **55** (1969) 137–165.

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